

MTH 605: Topology I

Practice Assignment IV

1. A space X is *weakly locally connected at x* if for every neighborhood U of x , there is a connected subspace of X contained in U that contains a neighborhood of x . Show that if X is weakly locally connected at every point, then X is locally connected.
2. Describe the components and path components of the following spaces.
 - (a) \mathbb{R}_ℓ
 - (b) \mathbb{R}^∞ with product and box topologies
3. Show that every compact subspace of a metric space is closed and bounded. Find a metric space in which the converse does
4. Show that if X is compact Hausdorff under two topologies τ and τ' , then either $\tau = \tau'$ or they are
5. Let Y be a compact space.
 - (a) Show that $\pi_1 : X \times Y \rightarrow X$ is a closed map.
 - (b) Let Y be a Hausdorff space, and let $f : X \rightarrow Y$. Then f is continuous if and only if the *graph* of f , $G_f = \{(x, f(x)) \mid x \in X\}$ is closed in $X \times Y$.
6. Show that a connected space having more than one point is uncountable.
7. Let $p : X \rightarrow Y$ be a surjective continuous map each of whose fibers is compact. Show that if Y is compact, then X is compact.
8. Let X be a compact Hausdorff space. Let \mathcal{B} be a collection of closed connected subsets that are simply ordered under inclusion. Then show that $\bigcap_{A \in \mathcal{A}} A$ is connected.
9. Establish the following facts.
 - (a) $[0, 1]$ is not compact in \mathbb{R}_K .
 - (b) \mathbb{R}_K is connected, but not path connected
 - (c) $[0, 1]$ is not limit point compact in \mathbb{R}_ℓ .
 - (d) Every subset of \mathbb{R} under the cofinite topology is compact.
 - (e) \mathbb{Q} is not locally compact.
10. A space X is *countably compact* if every countable covering of X has a finite sub-covering. Show that in a T_1 space X , countable compactness is equivalent to limit point compactness. [Hint: If not finite subcollection of U_n covers X , then choose $x_n \notin U_1 \cup \dots \cup U_n$ for each n .]
11. Let (X, d) be a compact metric space. Show that every isometry on X is a homeomorphism.
12. Let G be a topological group.

- (a) Show that if C is a component of G containing the identity element, then $C \trianglelefteq G$.
 - (b) If G is locally compact and $H \leq G$, then G/H is locally compact.
13. Show that a homeomorphism of locally compact Hausdorff spaces extends to their one-point compactification.
14. Describe the one-point compactification of the following spaces.
- (a) \mathbb{R}
 - (b) \mathbb{Z}_+
 - (c) \mathbb{R}^n